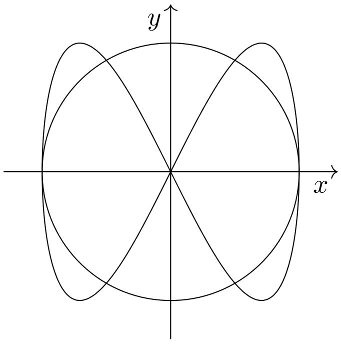
1901. If $y = 9^x$, write $(3^x + 1)^2(3^{-x} - 1)^2$ in terms of $y$.

1902. Find the linear function g which best approximates $\text{f}(x) = x^3 - 4x$ at $x = \sqrt{8}$.

1903. The diagram shows the circle $x^2 + y^2 = 1$ and the curve $x = \sin t, y = \sin 2t$. All values $t \in [0, 2\pi)$ are shown.




Determine whether the curve intersects the circle for any parameter values $t \in [0, \pi/6]$.

1904. Histograms are often set up with narrower classes around the centre of the data and wider classes in the tails. Explain why this is so.

1905. Sketch the following graphs, for large $k \in \mathbb{N}$:

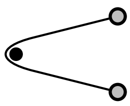
(a) $y = x^{2k}$,

(b) $y = x^{2k+1}$.

1906. Determine, in the form $y = \text{f}(x)$, all solution curves satisfying the following differential equation, in which $k$ is a constant:

$$\frac{d^3y}{dx^3} = k.$$

1907. A catapult fires ball-bearings of mass 40 grams from a light elastic sling attached to two fixed prongs. The prongs are 10 cm apart.



(a) Calculate the initial horizontal acceleration of the ball-bearing if it is drawn back a distance of 20 cm, generating a tension of 1.2 Newtons in the sling.

(b) State a simplifying assumption that you made to produce an answer to part (a).

1908. You are given that $\text{cosec}\, 75° = \sqrt{6} - \sqrt{2}$. Using the Pythagorean trig identity $1 + \cot^2 x \equiv \text{cosec}^2 x$, or otherwise, show that $\tan 75° = 2 + \sqrt{3}$.

1909. A set of five data is known to have mean, median and mode 0, and largest value 1. Show that the range $R_x$ of the data must satisfy $\frac{3}{2} < R_x < 3$.

1910. Show that $\dfrac{2^x - 1}{4^x - 1} \equiv \dfrac{1}{2^x + 1}$.

1911. In each case, decide which of the symbols $\Longrightarrow$ or $\Longleftrightarrow$ should occupy the space.

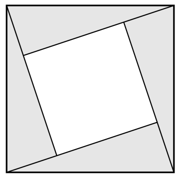
(a) $x = y \qquad \sin x = \sin y$,

(b) $x = y \qquad \arcsin x = \arcsin y$,

(c) $x = y \qquad |x| = |y|$.

1912. Simplify $\dfrac{3x^2 + 5xy + 2y^2}{6x^2 - 11xy - 10y^2}$.

1913. The figure below consists of four congruent right-angled triangles and two squares.



By calculating its area in two different ways, prove Pythagoras's theorem.

1914. Prove or disprove the following statement:

$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \iff \mathbb{P}(A' \mid B) = \mathbb{P}(A').$$

1915. In an attempted proof, a student writes: "If three linear equations in two unknowns have exactly one simultaneous solution, then two of the equations must be identical." Give a counterexample to show that this is not true.

1916. Show that $y = x^4 - 3x^3 + 3$ has a stationary point of inflection.

1917. $A_n$ is a quadratic sequence and $B_n$ is a non-constant arithmetic sequence. Determine whether the following sequences are quadratic, arithmetic or neither:

(a) $u_n = A_n + B_n$,

(b) $u_n = A_n B_n$.

1918. Two particles move along an $x$ axis. They have displacements given, for $t \geq 0$, by

$$x_1 = \frac{t^2}{t^2 + 1}, \qquad x_2 = \frac{4t}{4t + 1}.$$

(a) Verify that the particles start at the origin.

(b) Find the $x$ location at which they next meet.

(c) What happens in the long-term?

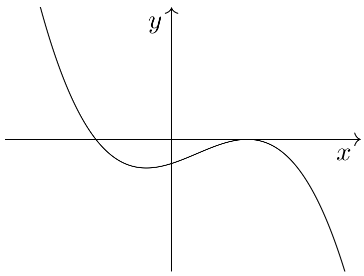
1919. Determine the number of real roots of

$$(x^2+3x+2)(x^4+3x^2+2)=0.$$

1920. Find the equation of the tangent to $y=1+\sin^2 x$ at $x=\frac{\pi}{4}$, in the form $ay+\pi=bx+c$, for $a,b,c\in\mathbb{Z}$.

1921. A point $(x,y)$ is randomly chosen inside the circle $x^2+y^2=1$. Find the probability that $y>|x|$.

1922. A cubic graph $y=\mathrm{f}(x)$ is shown below.




State, with a reason, whether the following are true descriptions of the function f:

(a) "one root of f is a single root",

(b) "one root of f is a double root",

(c) "one root of f is a triple root",

(d) "one root of f is a repeated root".

1923. Prove that every quadratic graph $y=ax^2+bx+c$ may be generated from $y=x^2$ by a combination of stretches and translations.

1924. Prove the following trigonometric identities, where $\theta$ is defined in degrees:

(a) $\sin(90°-\theta)\equiv\cos\theta$,

(b) $\tan(90°+\theta)\equiv-\cot\theta$.

1925. Functions f and g are defined as $\mathrm{f}(x)=10^x$ and $\mathrm{g}(x)=x^2-2x-8$.

(a) Show that $\mathrm{g}(x)$ is increasing on $(1,\infty)$,

(b) Write down the set of $x$ values for which $\mathrm{fg}(x)$ is increasing.

1926. If $\dfrac{d}{dx}\left(x^3+\dfrac{dy}{dx}\right)=1$, find $\dfrac{d^2y}{dx^2}$ in terms of $x$.

1927. On a $2\times 3$ grid, three red counters and three green counters are placed at random.

(a) Show that the probability that the two colours end up in distinct rows is 10%.

(b) Determine the probability that no two same-coloured counters are adjacent to each other.

1928. Show that the curves $x^2+3x+y^2+5x=10$ and $x^2+y^2=1$ do not intersect.

1929. "The line $y=0$ is a tangent to $y=x^4-x^2-12$." Is this statement true or false?

1930. Let the constants $a,b,c,d$ be distinct and positive, and let $k$ be a natural number. Find any $x$ values for which the following function is undefined:

$$x\mapsto\frac{(x^{2k}-a^{2k})(x^{2k}+b^{2k})}{(x^{2k}-c^{2k})(x^{2k}+d^{2k})}.$$

1931. Solve the following equation for $x\in[-\pi,\pi]$:

$$\operatorname{cosec}x+\operatorname{cosec}^2 x=2.$$

1932. Give a set of integers that is a counterexample to the claim: "The interquartile range is always smaller than the range."

1933. The equation of a straight line, gradient $m$, passing through the point $(a,b)$ is

$$\frac{y-b}{x-a}=m.$$

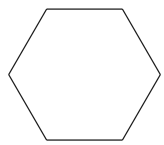
Sketch the following graphs:

(a) $\dfrac{y-b}{(x-a)^3}=m$

(b) $\dfrac{(y-b)^3}{x-a}=m.$

1934. Find the area enclosed by the graph $y=x^3+x+2$ and the coordinate axes.

1935. Two of the sides of a regular hexagon $ABCDEF$ are given by vectors $\overrightarrow{AB}=2\mathbf{i}$ and $\overrightarrow{BC}=\mathbf{i}+\sqrt{3}\mathbf{j}$.



Find the following vectors:

(a) $\overrightarrow{CD}$,

(b) $\overrightarrow{AD}$.

1936. Functions f and g have domains and codomains

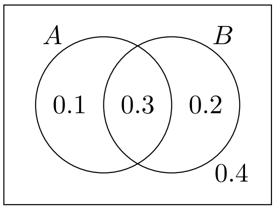
$$\mathrm{f}:A\mapsto B,$$
$$\mathrm{g}:C\mapsto D.$$

Each function is the other's inverse. State, with a reason, what must be true of sets $A,B,C,D$.

1937. Prove algebraically that, if a geometric iteration $u_{n+1}=ru_n$ has a fixed point $x$, then at least one of $x=0$ or $r=1$ must hold.

1938. Disprove the following claim: “If $\mathrm{f}(a) < 0 < \mathrm{f}(b)$, then there must be a root of f between $a$ and $b$.”

1939. A Venn diagram contains probabilities as shown:




Solve for $k$ in the equation $\mathbb{P}(A \mid B) = k\,\mathbb{P}(B \mid A)$.

1940. Give, by sketching, two different reasons why the Newton-Raphson method may fail to converge to a given root, despite starting close to it.

1941. State, giving a reason, which of the implications $\Longrightarrow$ , $\Longleftarrow$ , $\Longleftrightarrow$ links the following statements concerning real numbers $x$ and $y$:

① $|x| = |y|$,

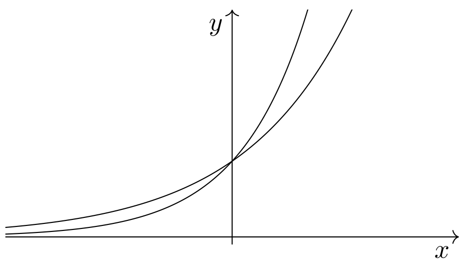
② $x^2 = y^2$.

1942. You are given that two functions f and g satisfy $\mathrm{f}(x) = |\mathrm{g}(x)|$ for all $x \in \mathbb{R}$. State, with a reason, whether the following statements are true or false:

(a) $\mathrm{f}'(x) = |\mathrm{g}'(x)|$ for all $x \in \mathbb{R}$,

(b) $\displaystyle\int_a^b \mathrm{f}(x)\,dx = \left|\int_a^b \mathrm{g}(x)\,dx\right|$ for all $x \in \mathbb{R}$.

1943. The graph shows $y = 2^x$ and $y = 3^x$:




Show that the graphs may be transformed to one another by a stretch in the $x$ direction.

1944. A sequence has ordinal definition $u_n = n^2 + n + 3$. Prove that $u_{n+1} < 2u_n$, for all $n \in \mathbb{N}$.

1945. Write down the equations of the reflections of the following graphs in the line $x = k$:

(a) $y = x - k$,

(b) $y = (x - k)^2$,

(c) $y = (x - k)^3$.

1946. Explain why one of the following expressions, in which $a$ is a non-zero constant, is well-defined and the other is not:

$$\left.\frac{t^3 - a^3}{t^2 - a^2}\right|_{t=a} \qquad \lim_{t \to a} \frac{t^3 - a^3}{t^2 - a^2}$$

1947. A dodecahedron has 12 faces, each of which is a pentagon. Two distinct faces of a dodecahedron are selected at random. Determine the probability that the two faces share a common edge.

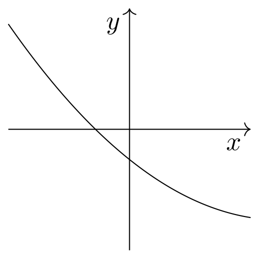
1948. Show that the sum of the first 100 natural numbers which are not divisible by 5 is 6250.

1949. In this question, do not use a calculator.

Given that $x = 3$ is a root, solve

$$\frac{4x}{x+1} - \frac{4x^2}{x-1} + 15 = 0.$$

1950. A section of the graph $y = \mathrm{f}(x)$ is as shown:




State whether, over the domain shown, the terms “increasing”, “decreasing”, “convex”, “concave” can be used to describe the function f.

1951. Write the following in terms of $e^x$:

(a) $e^{3x}$,

(b) $e^{3x-1}$,

(c) $e^{3x-2}$.

1952. Prove that $\displaystyle\lim_{x \to \infty} \frac{2x+5}{2x-1} = 1$.

1953. A student says: “The graph $y = 3 - 2|x|$ is always above the $x$ axis, because the modulus function makes everything positive.” Explain the error, and sketch the correct graph.

1954. Two perpendicular vectors are given as

$$\mathbf{a} = \begin{pmatrix} x \\ 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} x+3 \\ -2 \end{pmatrix}.$$

Find the possible values of $x$.

1955. A triangle, with initial lengths $(3, 4, 5)$, undergoes an enlargement: the perpendicular sides lengthen continuously, at a rate of 1 unit per second, while remaining perpendicular. Find the exact rate at which the hypotenuse is lengthening when the area of the triangle is 15 square units.

1956. The variable $Y$ has a normal distribution. You are given that $\mathbb{P}(Y < 0) = \mathbb{P}(Y > 4) = 0.2$.

(a) Write down the mean.

(b) Find the standard deviation.

1957. $\text{F}(x) = x^3 - 12x$ is not invertible over $\mathbb{R}$.

(a) Explain why not.

(b) By considering stationary points, determine the largest $a > 0$ such that F is invertible over the domain $[-a, a]$.

(c) Give the codomain for this version of F.

1958. A question is written as follows: "The inequality $ax^2 + bx + c > 0$ has solution set $(\infty, 4) \cup (5, \infty)$. Find the constants $a, b, c$."

(a) Show that it is not possible to determine $a, b, c$ with this information.

(b) Find the set of possible values of $a$.

1959. Sketch $4 \geq (x-2)^2 + (y-3)^2 \geq 9$.

1960. Find the values $x \in [0, 2\pi)$ for which the function $x \mapsto \sin x + \cos^2 x$ has a local maximum.

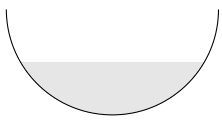
1961. The variables $Z_1$ and $Z_2$ are independent, and each follows the same normal distribution $Z \sim N(0, 1)$. Find the following probabilities:

(a) $\mathbb{P}(Z_1, Z_2 > 0)$,

(b) $\mathbb{P}(Z_1 < Z_2)$,

(c) $\mathbb{P}(0 < Z_1 < Z_2)$.

1962. A gutter is a half-cylinder of radius $r$.



Show that, when water stands to a depth of $\frac{1}{2}r$, the fraction of the volume of the gutter which is occupied by the water is $\frac{1}{6}\left(4\pi - 3\sqrt{3}\right)$.

1963. The generalised binomial expansion, which is valid for $|x| < 1$, has the following formula:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$$

(a) Find a quadratic approximation to $(1+4x)^{-1}$.

(b) Give the domain of validity of the expansion in the form $|x| < k$.

1964. By taking out a factor of $(3x-2)$, or otherwise, solve the equation $(3x-2)^3 + 8 = 12x$.

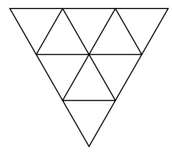
1965. One of the following statements is true; the other is not. Identify and disprove the false statement.

① $\sin^2\theta = \frac{3}{4} \implies \cos\theta = \frac{1}{2}$,

② $\sin^2\theta = \frac{3}{4} \impliedby \cos\theta = \frac{1}{2}$.

1966. Give the formula for the speed of the end-point of a radius of length $r$ sweeping out an angle of $\omega$ radians per second at the centre of a circle.

1967. In the diagram, a pattern consisting of nine small triangles is depicted:



Find the number of different ways of colouring the pattern if:

(a) two colours are used, and two small triangles sharing an edge cannot be the same colour,

(b) three colours are used, with no restrictions.

1968. Solve the inequality $x^2 - x + 6 > 0$, giving your answer in set notation.

1969. Show that $y = x^2 - x$ and $x = y^2 - y$ are tangent.

1970. A general recurring decimal is $x = 0.\dot{a}_1 a_2 ... \dot{a}_n$, where $a_1, a_2, ..., a_n$ represent digits. Prove that

$$x = \frac{a_1 a_2 ... a_n}{\underbrace{99...9}_{n\ 9\text{'s}}}.$$

1971. Shade the region of the $(x, y)$ plane which satisfies both of the following inequalities:

$$|x-2| < 1, \qquad |y-3| < 1.$$

1972. A particle moves in one dimension with velocity given by $v = 3t^2 + 2$ for $t \in [0, 4]$. Find the time at which the particle's instantaneous velocity is equal to its average velocity over the time period. Give your answer in exact form.

1973. Find the coefficient of $x^2$ in the polynomial expansion of $\left(1 + 3x + 2x^2\right)^5$.

1974. Solve, for $x, y \in [0, 360°)$,

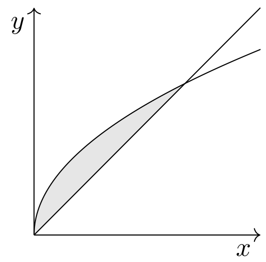
$$\begin{aligned} 2\sin x + \cos y &= 1 \\ \sin x - 4\cos y &= 5. \end{aligned}$$

1975. A statistician models the maximum running speed of adult humans with a normal distribution. Give two reasons why this distribution will not give an accurate picture of reality.

1976. Prove that, for $a \neq b$, $c \neq d$,

$$\frac{a}{b} = \frac{c}{d} \implies \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

1977. The curve $y = \sqrt{x}$ and $y = x$ enclose a region:




This area of this region is being approximated, by using the trapezium rule with four strips. Show that the percentage error in the evaluation of the area is approximately 14%.

1978. A differential equation has led to

$$\int \frac{1}{x^2}\,dx = \int \frac{1}{y^2}\,dy.$$

Show that the solution curves are $y = \dfrac{x}{kx+1}$.

1979. Disprove the following statement: "On an $(x, y)$ plane, a parabolic curve and a cubic curve must always intersect."

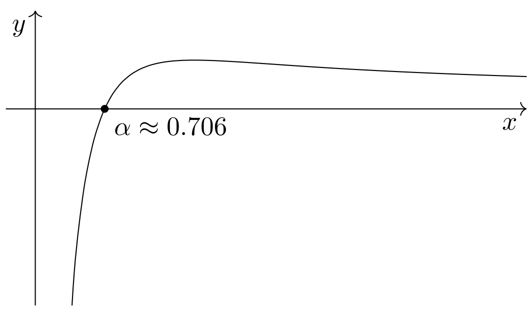
1980. Consider the function $f(x) = 4 \times 4^x - 5 \times 2^x + 1$, defined over $\mathbb{R}$.

(a) Show that the range is $\{y \in \mathbb{R} : y \geq -9/16\}$.

(b) Show that the roots of $f(x)$ differ by 2.

1981. If $\frac{d}{dx}\sqrt{x+y} = x$, find $\frac{dy}{dx}$ in terms of $x$ and $y$.

1982. The graph below shows $y = f(x) = e^{-x} + x^{-1}\ln x$. The equation $f(x) = 0$ has one real root at $\alpha$, and the graph has one stationary point.




(a) Find $f'\left(\frac{1}{2}(1+\sqrt{5})\right)$.

(b) Hence, explain how you know that, for any starting value $x_0 \geq \frac{1}{2}(1+\sqrt{5})$, the Newton-Raphson method will fail to converge to $\alpha$.

1983. Prove that $\cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$.

1984. In this question, the substitution $z = x^2 + 4$ is used to transform the definite integral

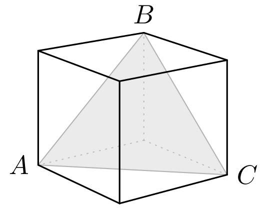
$$I = \int_0^1 6x\sin(x^2+4)\,dx.$$

(a) By differentiating, show that the relationship between infinitesimal changes in $x$ and $z$ may be expressed as $3\,dz = 6x\,dx$.

(b) Hence, show that $I = 3\displaystyle\int_4^5 \sin z\,dz$.

1985. If $f(x) = \dfrac{1}{1+x}$, prove that $f^3(x) = \dfrac{x+2}{2x+3}$.

1986. The diagram shows a cube of unit side length.




Find the area of triangle $ABC$.

1987. In a game of cards, player $A$ and player $B$ swap a card at random. Before the swap, $A$ has two Jacks and a Queen, and $B$ has two Queens and a King. Find the probability that, after the swap:

(a) $A$ has a pair,

(b) $B$ has at least a pair.

1988. You are given that a function g and its derivative g′ are related in the following way:

$$2\,g'(x)(g(x) - 1) = 1.$$

Verify that the function $g(x) = \sqrt{x} + 1$ satisfies this relationship.

1989. Prove the following implication:

$$\big(|x| - |y|\big)\big(|x| + |y|\big) = 0 \iff x^2 = y^2.$$

1990. Ramanujan produced the following extraordinary infinite series approximation to $\frac{1}{\pi}$:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}.$$

Using the fact that 0! is defined to be 1, show that the first term of this series gives an approximation of $\pi$ correct to six decimal places.

1991. Using integration, find the average value of the function $f(x) = \sqrt{x}$ on the domain $[0, 81]$.

1992. A variable has distribution $X \sim B(5, p)$. It is given that $\mathbb{P}(X < 1 \mid X < 2) = 1/6$. Find $p$.

1993. A particle follows a path such that, at time $t \geq 0$, it is at coordinates $\left(3t^2 - 4, t^3 - 1\right)$.

(a) Find, in terms of $t$, $\dfrac{dy}{dx}$ for the path.

(b) Hence, determine the particle's position when it is travelling parallel to the line $y = x$.

1994. Determine whether the curve $y = x^3 - 4x^2 + 2x - 1$ is concave, convex or neither in the vicinity of the point $(3, -4)$.

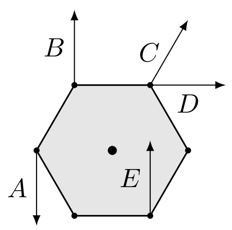
1995. Two sequences are defined ordinally by

$$a_n = 20n - n^2,$$
$$b_n = 500 - 40n + n^2.$$

Show that $|a_n - b_n|$ is minimised at $n = 15$.

1996. Either prove or disprove the following statement: "If $f'(x) = f(x)$, then $f''(x) = f'(x)$".

1997. A merry-go-round is a regular hexagon, which may rotate freely about its centre in a horizontal plane. Five children exert forces as depicted below, all of which have the same magnitude and are parallel or perpendicular to sides of the hexagon.




Determine, showing your reasoning carefully, the direction in which the merry-go-round will rotate.

1998. Solve for $y$ in the following equation, giving your answer in exact form:

$$\frac{\sqrt{y}}{1 + \sqrt{y}} + \frac{\sqrt{y} + 1}{\sqrt{y} - 1} = 4.$$

1999. A student writes: "When you press the accelerator of a resting car, a driving force is generated which acts forwards on the ground. It is the NIII pair of this force which causes the car to accelerate." Explain whether this is correct.

2000. A non-linear differential equation is given as

$$y^2 + \frac{dy}{dx} + 1 = 0.$$

Verify that $y = \cot x$ satisfies the DE.

——— End of Volume II ———